② how did we get $\phi(x\ y\ z) = f(z)\, e^{i(kx - \omega t)}$

why in this form?

1 The equations of motion certainly have many and varied solutions. We are *not* trying to find the *general* solution (i.e. a formula that covers all solutions).

2 Rather, we are trying to see if there are any solutions that look like "trains of waves": we are looking for disturbances that move at a uniform speed in a direction we will call the "x" direction, and have constant value (at any given moment) along horizontal lines perpendicular to the x-axis, i.e. parallel to the y-axis. A snapshot of the surface disturbance for such a motion might look something like this:

Here the x-axis is pointing more-or-less towards you, and the y-axis is parallel to the long edges of the photo.

3 Experience tells us that small-amplitude oscillatory solutions of linear equations are often sinusoidal in time, i.e. at any point in space

$$\phi(t) = \mathrm{Re}\left[f e^{-i\omega t}\right],$$

where f is a complex constant. We write it this way for convenience. To see the meaning, suppose we write f in polar form: $f = |f|\, e^{i\delta}$. Here $|f|$ is called the modulus of f (its "magnitude") and δ is called its "argument" or "phase". Then

$$\phi(t) = \mathrm{Re}\left[|f|\, e^{i\delta} e^{-i\omega t}\right] = \mathrm{Re}\left[|f|\, e^{i(\delta - \omega t)}\right] = |f| \cos(\omega t - \delta),$$

and we see that the ***mod*** of f specifies the amplitude of $\phi(t)$, and the ***arg*** of f specifies the phase-lag relative to $\cos(\omega t)$. (Have used Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$, and $\cos(-x) = \cos(x)$ here.)

4 To get this disturbance to “move”, i.e. to be a “fixed” disturbance that it translating uniformly in time along the x-axis, we make

$$\phi(x,t) = \mathrm{Re}\left[f e^{i(kx-\omega t)}\right].$$

You can see that the value of ϕ is constant if $kx-\omega t$ is held constant, i.e. if we let

$$x = \frac{\omega}{k}t + \mathrm{const}$$

i.e, if x increases uniformly with speed ω/k. Thus in a reference-frame that is moving steadily along the x-axis with speed ω/k, the disturbance field does not change in time. This is what we call a “travelling wave”, and this is the kind of solution we are looking for.

5 Now we suppose there is no y-dependence, and finally we do not restrict the z-dependence of the amplitude or phase of ϕ - just let f be an arbitrary function of z, and thus write

$$\phi(x,y,z,t) = \mathrm{Re}\left[f(z)e^{i(kx-\omega t)}\right].$$

And that’s where that form for ϕ came from! It is the general form of sinusoidal plane wave train, and the function $f(z)$ is initially unknown. (Later we discovered $f(z)$ had to be a multiple of e^{kz}.)