MTH 455 A DISCRETE LINEAR AGE-STRUCTURED MODEL

JR 2/1/2005 age_structured.mws
See Haberman, Section 35, p138.

```
> # SET UP THE MATRIX A
  restart:
  with(LinearAlgebra):
  M:=3;
  b:=[0,1.7,0.5,0];       # birth rates
  d:=[0.2,0.1,0.6,0.9]; # death rates
  A:=[seq([seq(0,col=0..M)],row=0..M)]:  #fill A with zeros
  for col from 0 to M-1 do
     A[1,col+1]:=b[col+1];
     A[col+2,col+1]:=1-d[col+1];
  end do:
  evalm(A);
```

$$M := 3$$

$$b := [0, 1.7, 0.5, 0]$$

$$d := [0.2, 0.1, 0.6, 0.9]$$

$$\begin{bmatrix} 0 & 1.7 & 0.5 & 0 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$$

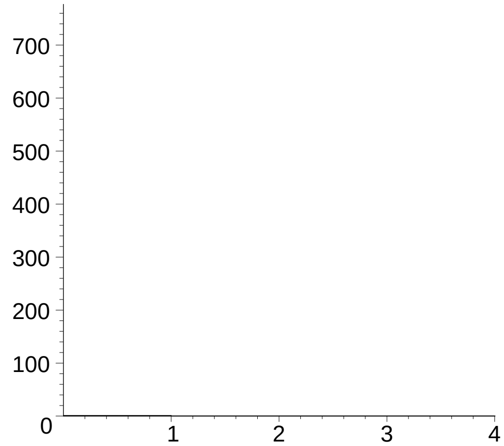
```
> # SOME USEFUL PROCEDURES

  # MUTLIPLY VECTOR N BY MATRIX A
  apply_A:=(A,N)->convert(evalm(A&*N),list):

  # NORMALIZE A VECTOR
  normaliz:=proc(N)
     local tot,i,Nlocal;
     Nlocal:=N;
     tot:=0;
     for i from 1 to nops(N) do tot:=tot+abs(N[i]); end do:
     unassign('i'):
     for i from 1 to nops(N) do Nlocal[i]:=N[i]/tot; end do:
  #   for i from 1 to nops(N) do Nlocal[i]:=N[i]/N[1]; end do:
     return Nlocal;
  end proc:

  # MAKE A HISTOGRAM OF A VECTOR
  my_histogram:=proc(N)
     local n,i,r;
     n:=nops(N);
     for i from 0 to n-1 do
       r[i]:=plottools[rectangle]([i,N[i+1]],[i+1,0],color=pink);
     end do:
     return plots[display](seq(r[i],i=0..n-1));
  end proc:
```

```
> N:=[1,0,0,0];    # WHAT HAPPENS IF WE START WITH JUST BABIES?
  for t from 1 to 30 do
     N:=apply_A(A,N);
  end do;
```

$$N := [1, 0, 0, 0]$$
$$N := [0., 0.8, 0., 0.]$$
$$N := [1.36, 0., 0.72, 0.]$$
$$N := [0.360, 1.088, 0., 0.288]$$
$$N := [1.8496, 0.2880, 0.9792, 0.]$$
$$N := [0.97920, 1.47968, 0.25920, 0.39168]$$
$$N := [2.645056, 0.783360, 1.331712, 0.103680]$$
$$N := [1.9975680, 2.1160448, 0.7050240, 0.5326848]$$
$$N := [3.94978816, 1.59805440, 1.90444032, 0.28200960]$$
$$N := [3.668912640, 3.159830528, 1.438248960, 0.761776128]$$
$$N := [6.090836378, 2.935130112, 2.843847475, 0.5752995840]$$
$$N := [6.411644928, 4.872669102, 2.641617101, 1.137538990]$$
$$N := [9.604346023, 5.129315942, 4.385402192, 1.056646840]$$
$$N := [10.91253820, 7.683476818, 4.616384348, 1.754160877]$$
$$N := [15.37010276, 8.730030560, 6.915129136, 1.846553739]$$
$$N := [18.29861652, 12.29608221, 7.857027504, 2.766051654]$$
$$N := [24.83185351, 14.63889322, 11.06647399, 3.142811002]$$
$$N := [30.41935546, 19.86548281, 13.17500390, 4.426589596]$$
$$N := [40.35882273, 24.33548437, 17.87893453, 5.270001560]$$
$$N := [50.30979070, 32.28705818, 21.90193593, 7.151573812]$$
$$N := [65.83896687, 40.24783256, 29.05835236, 8.760774372]$$
$$N := [82.95049153, 52.67117350, 36.22304930, 11.62334094]$$
$$N := [107.6525196, 66.36039322, 47.40405615, 14.48921972]$$
$$N := [136.5146966, 86.12201568, 59.72435390, 18.96162246]$$
$$N := [176.2696036, 109.2117573, 77.50981411, 23.88974156]$$
$$N := [224.4148945, 141.0156829, 98.29058157, 31.00392564]$$
$$N := [288.8719517, 179.5319156, 126.9141146, 39.31623263]$$
$$N := [368.6613138, 231.0975614, 161.5787240, 50.76564584]$$
$$N := [473.6552164, 294.9290510, 207.9878053, 64.63148960]$$
$$N := [605.3732893, 378.9241731, 265.4361459, 83.19512212]$$
$$N := [776.8891673, 484.2986314, 341.0317558, 106.1744584]$$

```
> # SAME EXPERIMENT, BUT MAKE AN ANIMATED HISTOGRAM THIS TIME
  N:=[1,0,0,0];
  histo[0]:=my_histogram(convert(N,list)):
  for t from 1 to 30 do
     N:=apply_A(A,N);
     histo[t]:=my_histogram(convert(N,list)):
  end do: unassign('t'):
  plots[display]( seq( histo[t], t=0..30 ), insequence=true );
```

$N := [1, 0, 0, 0]$




```
> # SAME EXPERIMENT, BUT THIS TIME NORMALIZE POPULATION
  # IN ORDER TO SEE BETTER WHAT'S HAPPENING TO THE AGE-DISTRIBUTION
  N:=[1,0,0,0];
  histo[0]:=my_histogram(convert(N,list)):
  for t from 1 to 30 do
     N:=apply_A(A,N);
     N:=normaliz(N);
     print(N);
     histo[t]:=my_histogram(convert(N,list)):
  end do: unassign('t'):
  plots[display]( seq( histo[t], t=0..30 ), insequence=true );
```

$N := [1, 0, 0, 0]$

[0., 1.000000000, 0., 0.]

[0.6538461538, 0., 0.3461538462, 0.]

[0.2073732719, 0.6267281105, 0., 0.1658986176]

[0.5934291581, 0.09240246406, 0.3141683778, 0.]

[0.3148796048, 0.4758180696, 0.08335048363, 0.1259518419]

[0.5438240983, 0.1610589891, 0.2738002816, 0.02131663092]

[0.3732849843, 0.3954247115, 0.1317476415, 0.09954266247]

[0.5106851299, 0.2066193389, 0.2462332948, 0.03646223628]

[0.4063580476, 0.3499735999, 0.1592962538, 0.08437209894]

[0.4894158941, 0.2358459888, 0.2285111713, 0.04622694531]

[0.4256419585, 0.3234758698, 0.1753657741, 0.07551639698]

[0.4760350714, 0.2542322272, 0.2173604783, 0.05237222322]

[0.4370861702, 0.3077507172, 0.1849026980, 0.07026041474]

[0.4677192116, 0.2656587970, 0.2104305220, 0.05619146939]

[0.4439496125, 0.2983198718, 0.1906222972, 0.06710821876]

[0.4625901424, 0.2727064934, 0.2061562491, 0.05854711522]

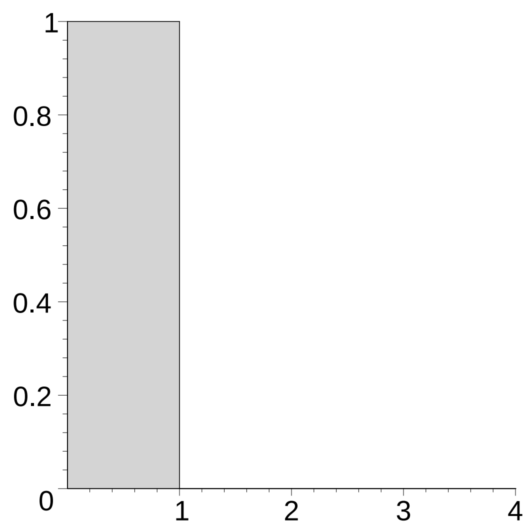
[0.4480918303, 0.2926281776, 0.1940741847, 0.06520580742]

[0.4594414011, 0.2770330818, 0.2035322682, 0.05999324898]

[0.4506012459, 0.2891800669, 0.1961653881, 0.06405329906]

[0.4575139372, 0.2796815505, 0.2019260300, 0.06087848226]

[0.4521249838, 0.2870863455, 0.1974351842, 0.06335348641]
[0.4563361400, 0.2812999249, 0.2009445212, 0.06141941331]
[0.4530515019, 0.2858132461, 0.1982072913, 0.06292796128]
[0.4556172094, 0.2822877855, 0.2003454055, 0.06174959922]
[0.4536153541, 0.2850384735, 0.1986771736, 0.06266899865]
[0.4551786608, 0.2828903824, 0.1999799441, 0.06195101324]
[0.4539586758, 0.2845667256, 0.1989632783, 0.06251132004]
[0.4549112524, 0.2832578200, 0.1997571012, 0.06207382685]
[0.4541677856, 0.2842793951, 0.1991375383, 0.06241528167]
[0.4547482380, 0.2834818128, 0.1996212544, 0.06214869498]



THE RELEVANCE OF THE EIGENDATA OF MATRIX A

```
> Am:=convert(A,Matrix); # "Eigenvectors" needs input of type
  "Matrix"
  #Eigenvalues(Am);
  (vals,vecs):=Eigenvectors(Am):
  eigvals:=vals;
  eigvecs:=evalm(vecs);
```

$$Am := \begin{bmatrix} 0 & 1.7 & 0.5 & 0 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$$

$$eigvals := \begin{bmatrix} 1.28102496759066553 + 0.\ I \\ -0.99999999999999944 + 0.\ I \\ -0.281024967590665364 + 0.\ I \\ 0.216769394122265754\ 10^{-17} + 0.\ I \end{bmatrix}$$

eigvecs :=

$[0.7902556456 + 0.\ I, -0.6679527490 + 0.\ I, 0.06194478282 + 0.\ I, -0.1191336664\ 10^{-15} + 0.\ I]$

$[0.4935145937 + 0.\ I, 0.5343621992 + 0.\ I, -0.1763395853 + 0.\ I, -0.2698502069\ 10^{-16} + 0.\ I]$

$[0.3467248068 + 0.\ I, -0.4809259793 + 0.\ I, 0.5647385290 + 0.\ I, 0.4181857162\ 10^{-16} + 0.\ I]$

$[0.1082648084 + 0.\ I, 0.1923703917 + 0.\ I, -0.8038268398 + 0.\ I, 1.000000000 + 0.\ I]$

```
> C1:=Column(vecs,1);  # THE DOMINANT EIGENVECTOR
  # COMPARE WITH SIMULATION (SCALE SO FIRST ELEMENT IS 1)
  C1/C1[1];
  N/N[1]; # N STORES THE POPULATION AT THE END OF OUR SIMULATION
```

$$C1 := \begin{bmatrix} 0.790255645579807986 + 0.\ I \\ 0.493514593749791064 + 0.\ I \\ 0.346724806785138617 + 0.\ I \\ 0.108264808432970413 + 0.\ I \end{bmatrix}$$

$$\begin{bmatrix} 1.00000000036549720 + 0.\ I \\ 0.624499928207509547 + 0.\ I \\ 0.438750180212220675 + 0.\ I \\ 0.136999727971708895 + 0.\ I \end{bmatrix}$$

$$[1.000000000, 0.6233818829, 0.4389709244, 0.1366661590]$$

OBSERVE THAT THE AGE DISTRIBUTION OF THE POPULATION IS SETTLING INTO THAT GIVEN BY THE DOMINANT EIGENVECTOR OF MATRIX A.

```
> ?Matrix
> ?Vector
> ?Column
>
```