MTH 455 EXAM #3

Answer 3 of the 4 problems.

1 Spatially homogeneous chemical oscillation?

1. We've seen oscillatory behavior in the Brusselator, but it was always associated with strong spatial variations. Can we get oscillation in a *spatially homogeneous* chemical reaction? Consider the "Lotka reaction":

$$\mathrm{A} + \mathrm{X} \xrightarrow{k_1} 2\mathrm{X}$$

$$\mathrm{X} + \mathrm{Y} \xrightarrow{k_2} 2\mathrm{Y}$$

$$\mathrm{Y} \xrightarrow{k_3} \mathrm{B}$$

Assume that the concentration of species A is kept constant.
(a) Using a for [A], x for [X], and y for [Y], write down the differential equations for x and y that come from the Principle of Mass Action.
(b) Clean up the equations using the change of variables

$$x = \frac{k_3}{k_2}u, \quad y = \frac{k_1 a}{k_2}v, \quad t = \frac{\tau}{k_1 a},$$

and finally the parameter shorthand, $\alpha = \frac{k_3}{k_1 a}$. Show that the equations become

$$\frac{du}{d\tau} = u(1-v), \qquad \frac{dv}{d\tau} = \alpha(u-1)v.$$

(c) Sketch the nullclines of this system in the 1st qudrant of the uv-plane, label them clearly, and decorate them with little line segments showing the required horizontal and vertical passage of solution curves. Do some spot checks to determine the direction of the phase-flow in each region, and use these and the nullclines to sketch a representative collection of plausible solution curves in the uv-plane.
(d) Choose one of the curves in your phase portrait in (c), and sketch the corresponding graph of u vs. τ.

2 Sinusoidal modes of diffusion equation

(a) Write down the diffusion equation in 1 spatial dimension (coordinate x), using u for the concentration of the stuff that's diffusing, t for time, and D for the diffusivity.
(b) Consider the possibility of solutions of the diffusion equation of the form

$$u(x,t) = \alpha e^{\omega t} \sin kx.$$

Determine if there is a relationship between k and ω that will make these functions solutions.
(c) If there are solutions of this type, for which k do they grow in amplitude and for which k do they decay? Is the growth/decay oscillatory or monotonic? For which k do they grow or decay fastest?
(d) If $D = 1$, and the initial condition is

$$u(x,0) = 2 + \sin x + \frac{1}{4} \sin 16x,$$

what is $u(x, \frac{1}{2})$? Make a rough sketch of $u(x,0)$ and $u(x, \frac{1}{2})$.

3 Linearization of Gray-Scott

(a) Demonstrate clearly that the Gray-Scott reaction-diffusion system

$$\frac{\partial U}{\partial t} = -UV^2 + F[1-U] + D_1 \frac{\partial^2 U}{\partial x^2},$$

$$\frac{\partial V}{\partial t} = UV^2 - (F+k)V + D_2 \frac{\partial^2 V}{\partial x^2},$$

in the case when $F = \frac{4}{100}$, $k = \frac{6}{100}$, has a uniform steady state $U(x,t) = \frac{1}{2}$, $V(x,t) = \frac{1}{5}$.
(b) Linearize (and clean up as much as possible) the PDEs at this uniform steady state.

4 Brusselator

In the 1D Brussellator, deviations from the uniform steady state $X = A$, $Y = B/A$, of the form

$$\xi(x,t) = \alpha e^{\omega t} \sin kx, \qquad \eta(x,t) = \beta e^{\omega t} \sin kx,$$

where $k = n\pi/L$, and L is the length of the reactor, are solutions of the reaction diffusion equations if a nontrivial solution can be found for the system

$$\begin{bmatrix} \omega - B + 1 + D_1 k^2 & -A^2 \\ B & \omega + A^2 + D_2 k^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In the case that $A = 1$, $B = 3$, $D_1 = \frac{1}{10}$, $D_2 = 1$, and $L = \pi$, find out whether the $n = 1$ modes and the $n = 5$ modes grow or decay, and whether monotonically or oscillatorily. Note: I am not asking you to compute the actual values of ω, only as much about them as you need to know in order to answer the question.